2D Image Segmentation Through the One Dimensional Mumford-Shah Functional

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Abstract. Image segmentation is a complex task, since it involves the manipulation of different kind of objects such as functions, domains in $\mathbb{R}^2$ and curves. In this work we propose a two-dimensional image segmentation method, using the Mumford and Shah's one dimensional technique, performing vertical and horizontal sweeps. This is in contrast to the technique that involves the generalization of the Mumford and Shah functional theory for 2D images.

Keywords: Mumford and Shah functional, Variational Method, Image Segmentation.

1 Introduction

There exist different methods for image segmentation, such as Canny and Sobel, among others. The traditional segmentation methods work by finding the edges of the different objects that are present on the image. The Mumford and Shah method, in addition to this, aims to find a smoothed version of the original image, which makes more attractive its study.

The aforementioned segmentation method, consists on finding a smoothed function and a set of edges (boundary) such that minimize a functional, which is constituted by the combination of the difference between the real image and the smoothed one, the gradient of the smoothed image and the boundary length. There exist developments of the Mumford and Shah methods for image segmentation in one (signals) and two dimensions (images) $[5, 6, 3, 2, 7]$.

Segmenting in one dimension, implies performing an horizontal cut on the image, so that we work only with a signal in $\mathbb{R}$, which facilitates both the mathematical and computational theories. In [5], the set of edges is determined, i.e., the points where there are intensity changes, and the function between these two points is found. Those are determined by minimizing the functional proposed by the aforementioned authors, analyzing it as a variational problem. First, the set of edges is fixed, and the smoothed function is varied so, the analytical expression of that function can be found. Then, the set of edges is varied, and in this way, the conditions to determine them are found. With this process, we obtain the segmentation of a tiny fraction of the image, in other words, we obtain a smoothed function through a straight line (horizontal cut), and the discontinuity points (edges) of such function are detected.

The study in [6] is the generalization of the functional proposed in [5], where the one variable functions are changed for two variable functions, with the objective of performing 2D image segmentation. The theory developed for that generalization is on $\mathbb{R}^2$, which increases its difficulty with respect to the theory developed on $\mathbb{R}$. To understand the 2D method, at least it is necessary to have knowledge of measure theory and distributions to understand the special functions of bounded variation and the theory that involves them [3]. As it is mentioned in [2], to avoid to work with those kind of functions, it can be treated only within the space of the L^2 functions, and using a weak norm to measure the approximation between functions. Moreover, it is required certain smoothness on the smoothed image f, therefore, f must belong to a Sovolev space. Another term is the measure of the edges, for this, it is employed the Hausdorff measure of dimension 1.

However, for one dimension it is enough to have knowledge on differential and integral calculus, differential equations and variational calculus. With this contrast, we can ask the question: can the same results be obtained as with the Mumford and Shah method on two dimensions, when the method on one dimension is used?

The objective of this article is targeted at answering this question. We are going to develop the theory for one dimension, to perform image segmentation of 2D images, without going through the mathematical sophistication that requires the generalization of the theory of the functional. This method is applied to every horizontal and vertical cuts of the image, to later, combine both results by averaging the information.

This paper is organized as follows. First, we describe the Mumford and Shah functional, as in [5], followed by its solution via the variational method. Later, the computational algorithm is presented, followed by the experimental results. At the end, the conclusions are stated.

2 The Mumford and Shah Functional

Let μ and ν be fixed parameters. Let $g : [a, b] \to \mathbb{R}$ be a given continuous function. For each k, the partition $\mathcal{P}[a, b]$ is given by

$$a = a_0 < a_1 < ... < a_k < a_{k+1} = b.$$

Let $X = \{f : [a, b] \to \mathbb{R} | f \in \mathcal{C}^1([a_i, a_{i+1}])\}$. We define $E : X \times \mathcal{P} \to \mathbb{R}$ as

$$E(f, \{a_i\}) = \mu^2 \int_a^b (f - g)^2 \, dx + \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} (\frac{df}{dx})^2 \, dx + \mu\nu k. \tag{1}$$

The first term is the approximation from the input signal to the signal that we are looking for, the second term is a measure of the smoothness on (a_i, a_{i+1}) of the signal that we are looking for, and the last term is the cardinality of the set of points $\{a_i\}$ on the open interval (a, b), which is equal to k.

2.1 Variational Method

We assume that f and $\{a_i\}$ minimize E. The first step is to vary f, to find its value through of the first variation of E and making it equal to zero, [4]. From this, we get the following elyptical problem with boundary values

$$f'' = \mu^2(f - g), \tag{2}$$

$$f'(a_i^{\pm}) = 0. \tag{3}$$

The solution to this problem can be found with the method of parameter variation, and can be written using the Green's function as follows

$$f(x) = f_\mu(x) + c_{1,i}K_\mu(x - a_i) + c_{2,i}K_\mu(a_{i+1} - x) \tag{4}$$

where

$$f_\mu(x) = \frac{\mu}{2}\int_{a_i}^{a_{i+1}} g(s)e^{-\mu|x-s|}\, ds \tag{5}$$

and

$$K_\mu(x) = \frac{\mu}{2}e^{-\mu|x|}, \tag{6}$$

$$c_{1,i} = \frac{2}{\mu^2(1-\alpha_i^2)}(f'_\mu(a_i) - \alpha_i f'_\mu(a_{i+1})), \tag{7}$$

$$c_{2,i} = \frac{2}{\mu^2(1-\alpha_i^2)}(-f'_\mu(a_{i+1}) + \alpha_i f'_\mu(a_i)), \tag{8}$$

$$\alpha_i = e^{-\mu(a_{i+1}-a_i)}. \tag{9}$$

The function K_μ is known as the Green's function [1]. Moreover, it can be seen that the function f_μ satisfies the differential equation (2).

The next step is to vary one of the points $\{a_i\}$, to obtain their values. After getting the first variation of E and using the fact that f_μ satisfies the equation (2), we get the condition

$$f''_\mu(a_i) \sim 0. \tag{10}$$

Then, the boundary that minimizes the energy E, for a large enough μ, will be the zeros of the second derivative of f_μ.

3 Computational Method

The implementation was done in MATLAB. First, the image was normalized by substracting the minimum and dividing it by the maximum minus the minimum, in order to achieve consistency in the dynamic range of the pixel intensities. The calculation of the integrals was done with MATLAB's function for the trapezoid rule. The calculation of the derivatives was done with the finite differences method. The algorithm for this is as follows.

Algorithm 1 Program for the Mumford and Shah method for image segmentation in 2D

1: $element = row$
2: $a = 1$
3: g equal to $element$ number a ,
4: $n = length(g)$,
5: Partition $x_j = 1, .., n$
6: Step size $h = 1$.
7: Calculate points $\{a_i\}$ with $i = 1...k$ such that (10) is satisfied, or equivalently, $f_\mu - g = 0$.
8: Save the values j_i with $i = 1, ..., k$ such that satisfy $0 \leq j_1 < ... < j_k < n$ and $a_i \in [x_{j_i}, x_{j_i+1}]$.
9: For each interval $[x_{j_i+1}, x_{j_{i+1}}]$, set $a_i = x_{j_i+1}$ and $a_{i+1} = x_{j_{i+1}}$ and calculate (6), (7), (8), (9), (5), (4) y (1).
10: Find the minimum energy value with the "hill climbing" method: vary j_i while the rest is fixed, and repeat step 9. Save the value j_i when the minimum energy is achieved, and repeat for each j_i.
11: $a = a + 1$
12: Repeat steps 3 to 10 until $a =$ number of rows of the image.
13: $element = column$
14: Repeat steps 3 to 10 until $a =$ number of columns of the image.
15: Calcular the average of steps 12 and 14 for f y j_i.

4 Experimental Results

We applied the previously described algorithm to two images, the first is the image known as Barbara, and the second is an image of the famous singer John Lennon. In both cases, we first used the parameters $\mu = 1$ and $\nu = 1$. The results are shown, first the smoothed image after applying the method to each row of the original image, then the smoothed image after applying the method to each column, and lastly the smoothed image consisting on the combination of the previous two. The execution time for each test image is presented. After that, the parameters for the image of John Lennon are modified, and the results are shown.

For the first example, the execution time was 108.31862 min. In Fig. 1 the Barbara input image is shown. The image size is 512×512. In Fig. 2, the smoothed images are shown, ordered as described in the previous paragraph. In Fig. 3, the edges that were obtained of each smoothed image are shown.

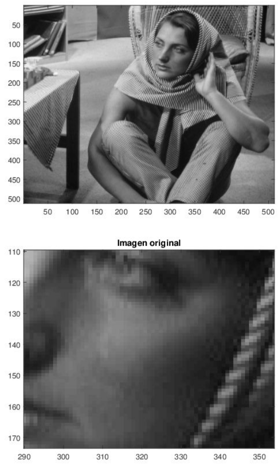

Fig. 1. Barbara original image (512×512)

Fig. 2. From left to right: Smoothed image of Barbara after using the method on each row of the original image, smoothed image after using the method on each column of the original image, resulting smoothed image after combining the previous two

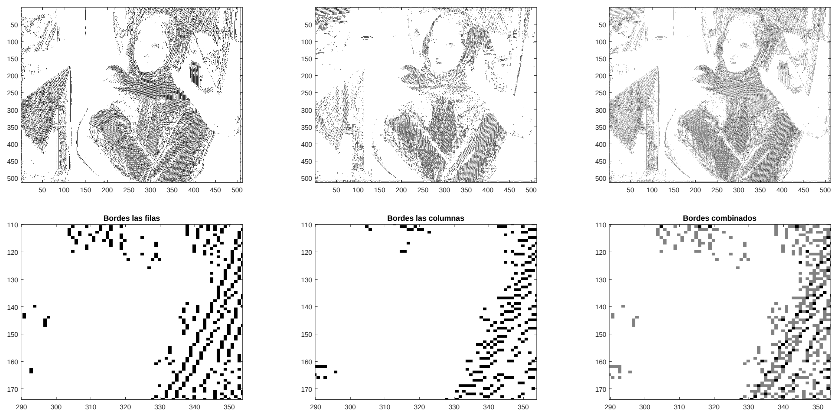

Fig. 3. From left to right: Edges obtained for the Barbara image after using the method on each row of the image, edges obtained after using the method on each column of the image, edges obtained after combination of the two previous results

For the second example, the execution time was 5.49119 min. In Fig. 4, the Lennon input image is shown, which is 204×204 pixel size. In Fig. 5, the smoothed images are shown in the same order as in the previous test. In Fig. 6, the edges obtained from the previous figures are shown.

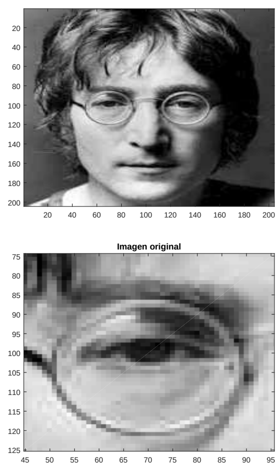

Fig. 4. Original image of John Lennon face (204×204)

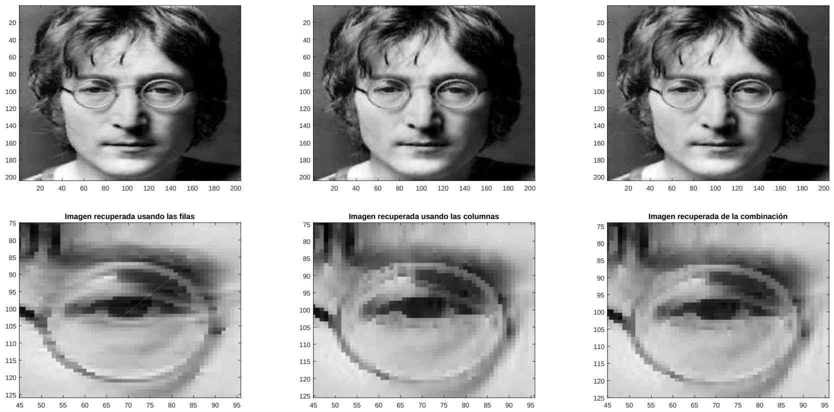

Fig. 5. From left to right: Smoothed John Lennon image after using the method on each row of the original image, smoothed image after using the method on each column of the original image, resulting smoothed image after combining the two previous images

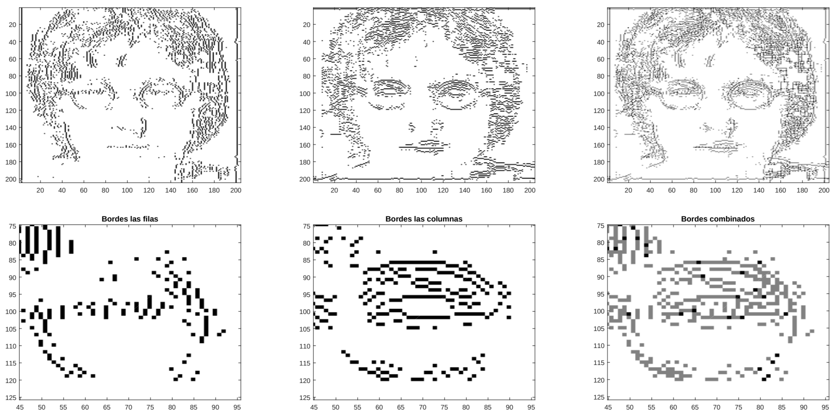

Fig. 6. From left to right: Edges obtained after using the method on each row of the image, edges obtained after using the method on each column of the image, edges obtained from the combination of the previous two

We present the John Lennon image changing the parameters. In Fig. 7 we show the smoothed images resulting from the combination of using the method of rows and columns when $\nu = 1$ and μ takes values of 0.5, 1 and 1.5. In Fig. 8 we show the edges that correspond to the smoothed images with the same values of μ and ν. In the same way, in Fig. 9 we present the smoothed images when we fix $\mu = 1$, and ν is varied with values of 0.5, 1 and 3. In Fig. 10 we show the corresponding edges.

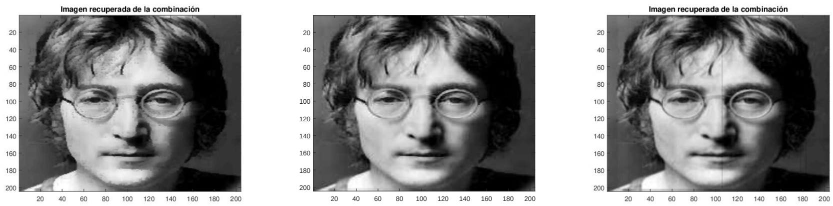

Fig. 7. Smoothed image of John Lennon when: μ=0.5 and ν=1 (left), μ=1 and ν=1 (center), μ=1.5 y ν=1 (right)

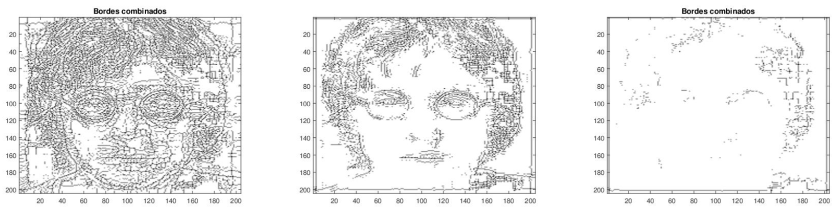

Fig. 8. Edges of the image of John Lennon when: μ=0.5 and ν=1 (left), μ=1 y ν=1 (center), μ=1.5 y ν=1 (right)

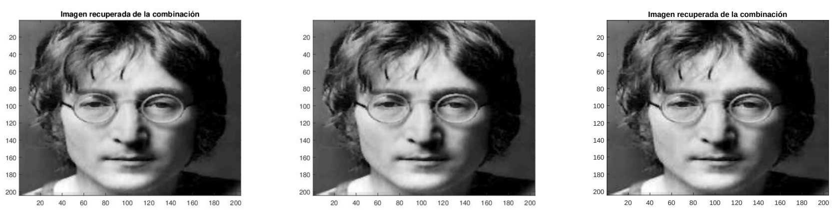

Fig. 9. Smoothed image of John Lennon when: μ=1 and ν=0.5 (left), μ=1 and ν=1 (center), μ=1 and ν=3 (right)

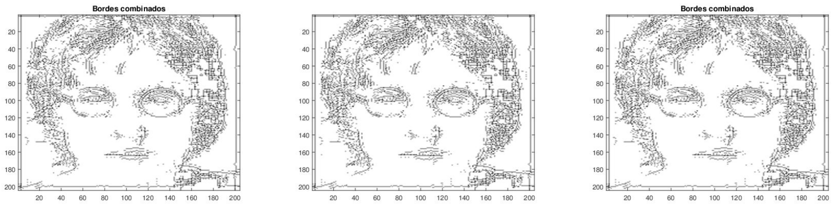

Fig. 10. Edges of the John Lennon image when: μ=1 and ν=0.5 (left), μ=1 y ν=1 (center), μ=1 y ν=3 (right)

It can be seen that the quality of the segmentation varies with respect to the value of the parameters μ and ν. It can be also noted that this variation is more sensitive to the variation of μ than it is to the variation of ν.

5 Conclusions

We achieved the implementation of a computation algorithm for 2D image segmentation based on the Mumford and Shah method for one dimension, i.e., the procedure for one dimension was generalized to two dimensions. The advantage of this technique, is that the theory is kept on $\mathbb{R}$, which is much less complex than working directly on $\mathbb{R}^2$.

The obtained results by applying this segmentation method, are the smoothed image that approximates to the input image, and the edges of that image. This methods has a high computational cost, since the execution time is large, which depends on the size of the image and number of edges. Despite of this, a good representation is obtained, although it depends on the parameters μ and ν.

The method still needs to be improved for it to be compared to the method designed for 2D, however, is a good start. In [7], the segmentation done for the Barbara image, with the approximation of Ambrosio and Tortorellini, can be found.

In a future study, we will make an optimization to the algorithm to decrease the execution time, in addition to combine this with other tools to improve the segmentation. We will also analyze to measure the performance of the method with noisy images, with the Peak signal-to-noise ratio (PSNR) metric[8] and Structural SIMilarity (SSIM) index[9].

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